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Euler constants for arithmetical progressions

by

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Dedicated to the memory of Yu. V. Linnik

1. Introduction. Euler's constant γ is defined by

$$\gamma = \lim_{m\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}-\log m\right\} = .5772156649.$$

In this paper we study the properties of the corresponding limit $\gamma(r, k)$ obtained by considering the sum of the reciprocals of the terms of the arithmetic progression

$$r, r+k, r+2k, \ldots \quad (0 < r \leqslant k).$$

In § 2, $\gamma(r, k)$ is defined precisely and shown to exist. In § 3 we show that $\gamma(r, k)$ differs from γ/k by a linear combination of logarithms of cyclotomic integers in the field of kth roots of unity. From this a formula for $\gamma(r, k)$ involving only real numbers is deduced and specialized for certain small k. In § 4 a study is made of the $\varphi(k)$ primitive $\gamma(r, k)$ in which r and k are coprime. In § 5 the connection is made between $\gamma(r, k)$ and the logarithmic derivative ψ of the Gamma function at the point r/k. The results of § 3 are now seen to give a really elementary proof of Gauss' theorem on $\psi(z)$ for rational z. In § 6 we make applications of $\gamma(r, k)$ to certain infinite series. In particular we develop the connection between $\gamma(r, k)$ and the class number of the quadratic fields $Q(\sqrt{\pm k})$. The final § 7 contains some comments on the numerical evaluation of $\gamma(r, k)$.

2. Definition and existence of $\gamma(r, k)$. We denote by $H(x, r, k)$ the general harmonic sum associated with the arithmetical progression $r, r+k, r+2k, \ldots$ namely

$$H(x, r, k) = \sum_{\substack{0<n\leqslant x\\ n\equiv r \pmod{k}}} \frac{1}{n}.$$

By this definition

$$H(x, r\pm k, k) = H(x, r, k).$$

We now define $\gamma(r, k)$ by

(1) $$\gamma(r, k) = \lim_{x\to\infty}\left\{H(x, r, k) - \frac{1}{k}\log x\right\},$$

so that

$$\gamma(0, 1) = \gamma.$$

Also

$$\gamma(r\pm k, k) = \gamma(r, k).$$

In other words, $\gamma(r, k)$ is a periodic function of r of period k.

Since

$$H(x, 0, k) = \frac{1}{k}H\left(\frac{x}{k}, 0, 1\right),$$

$$\gamma(0, k) = \frac{1}{k}\lim_{x\to\infty}\left\{H\left(\frac{x}{k}, 0, 1\right) - \log\frac{x}{k}\right\} - (\log k)/k$$

or

(2) $$\gamma(0, k) = (\gamma - \log k)/k.$$

To see that $\gamma(r, k)$ exists for $r \not\equiv 0 \pmod k$ we can note that, for $0 < r < k$,

$$U_n \equiv \frac{1}{r+nk} - \frac{1}{k}\log\frac{r+(n+1)k}{r+nk} = \int_0^1 \frac{kt}{(r+nk)(r+nk+t)}\,dt = O(n^{-2}).$$

Hence the infinite series

$$\sum_{n=0}^{\infty} U_n = \lim_{x\to\infty}\sum_{n=0}^{[x/k]} U_n = \lim_{x\to\infty}\left\{H(x, r, k) - \frac{1}{k}\log x\right\}$$

converges to a limit which we call $\gamma(r, k)$.

From the definition (1) we see that

(3) $$\sum_{r=0}^{k-1}\gamma(r, k) = \gamma$$

and more generally

(4) $$\sum_{\lambda=0}^{k-1}\gamma(r+\lambda m, mk) = \gamma(r, k).$$

In particular ($m = 2$) we have

(5) $$\gamma(r, 2k) + \gamma(r+k, 2k) = \gamma(r, k).$$

Also from the definition, $\gamma(r, k)$ is a strictly decreasing function of r. More precisely

$$\gamma(1, k) > \gamma(2, k) > \ldots > \gamma(k-1, k) > \gamma(k, k) = \gamma(0, k).$$

3. A closed form for $\gamma(r, k)$. We begin with

THEOREM 1. *For $k > 1$,*

(6) $$k\gamma(r, k) = \gamma - \sum_{j=1}^{k-1} e^{-2\pi rij/k}\log(1 - e^{2\pi ij/k}).$$

Proof. For $r = 0$ this is easy. The right hand side of (6) becomes

$$\gamma - \sum_{j=1}^{k-1}\log(1 - e^{2\pi ij/k}) = \gamma - \log\left\{\prod_{j=1}^{k-1}(1 - e^{2\pi ij/k})\right\} = \gamma - \log F(1)$$

where

$$F(x) = \prod_{j=1}^{k-1}(x - e^{2\pi ij/k}) = \frac{x^k - 1}{x - 1}.$$

Since $F(1) = k$, the theorem for $r = 0$ now follows from (2).

Suppose now that $r \not\equiv 0 \pmod k$. For simplicity we write ε for $e^{2\pi i/k}$. Consider the finite Fourier series generated by $\gamma(r, k)$ namely

(7) $$\sigma_j = \sum_{\lambda=0}^{k-1}\gamma(\lambda, k)\varepsilon^{\lambda j}.$$

By (3),

$$\sigma_0 = \gamma.$$

When $j \neq 0$

$$\sigma_j = \lim_{x\to\infty}\left\{\sum_{\lambda=0}^{k-1}\left(H(x, \lambda, k) - \frac{1}{k}\log x\right)\varepsilon^{\lambda j}\right\}.$$

Since

(8) $$\sum_{\lambda=0}^{k-1}\varepsilon^{\lambda j} = 0,$$

we have

(9) $$\sigma_j = \lim_{x\to\infty}\sum_{\lambda=0}^{k-1}H(x, \lambda, k)\varepsilon^{\lambda j} = \sum_{n=1}^{\infty}\frac{\varepsilon^{jn}}{n} = -\log(1 - \varepsilon^j).$$

Multiplying both members of (7) by ε^{-jr} and summing over j gives us

$$\sum_{j=0}^{k-1}\sigma_j\varepsilon^{-jr} = \sum_{\lambda=0}^{k-1}\gamma(\lambda, k)\sum_{j=0}^{k-1}\varepsilon^{j(\lambda-r)} = k\gamma(r, k)$$

in view of (8). Substituting for σ_j from (9) gives the theorem.

The simplest instances of Theorem 1 are for $k = 2$. In this case $1-e^{2\pi ij/k} = 2$ so that we have

$$\gamma(0,2) = \tfrac{1}{2}(\gamma-\log 2) = -.05797\ldots,$$
$$\gamma(1,2) = \tfrac{1}{2}(\gamma+\log 2) = .63518\ldots$$

These results also follow at once from (3) and (2). For $k > 2$ the terms of the sum in (6) become complex. Since $\gamma(r,k)$ is real we can replace the sum by its real part. However, this leaves something to be desired, namely a simplification using real logarithms. To achieve this we prepare

LEMMA A.

(a) $$\sum_{j=1}^{k-1}\sin\frac{2\pi rj}{k} = 0,$$

(b) $$\sum_{j=0}^{k-1}\cos\frac{2\pi rj}{k} = 0 \quad \textit{if} \quad r \not\equiv 0 \pmod{k},$$

(c) $$\sum_{j=0}^{k-1} j\sin\frac{2\pi rj}{k} = \begin{cases} 0 & \textit{if} \quad r \equiv 0 \pmod{k}, \\ -\dfrac{k}{2}\cot\dfrac{\pi r}{k} & \textit{otherwise}. \end{cases}$$

Proof. The well known sums (a) and (b) are the imaginary and real parts of the geometric progression sum

$$\sum_{j=0}^{k-1} e^{2\pi irj/k} = (e^{2\pi ir}-1)/(e^{2\pi ir/k}-1) = 0.$$

One way to prove (c) is via the identity

$$\sum_{j=0}^{k-1} ju^j = (1-u)^{-2}\{ku^k(u-1)-u(u^k-1)\}$$

which is established by an easy induction on k. When u is a kth root of unity ($\neq 1$) this becomes

$$\sum_{j=0}^{k-1} ju^j = -\frac{k}{1-u},$$

or

$$\sum_{j=0}^{k-1} j(u^j-u^{-j}) = -k\frac{1+u}{1-u}.$$

For $r \not\equiv 0 \pmod{k}$ and $u = \varepsilon^r$ this becomes

$$\sum_{j=0}^{k-1}\sin\frac{2\pi rj}{k} = \frac{1}{2i}(-k)\frac{1+\varepsilon^r}{1-\varepsilon^r} = -\frac{k}{2}\cot\frac{\pi r}{k}.$$

This is (c).

We return now to the sum in Theorem 1. We may assume that $r \not\equiv 0 \pmod{k}$. Next we observe that

$$1-\varepsilon^j = -2ie^{\pi ij/k}\sin\frac{\pi j}{k}$$

so that

$$\log(1-\varepsilon^j) = \log 2+\log\sin\frac{\pi j}{k}+\frac{\pi i}{2k}(2j-k)$$

and

$$\varepsilon^{-rj} = \cos\frac{2\pi rj}{k} - i\sin\frac{2\pi rj}{k}.$$

Hence Theorem 1 is equivalent to

$$(10)\quad \gamma-k\gamma(r,k) = \operatorname{Re}\Big\{\sum_{j=1}^{k-1}\varepsilon^{-rj}\log(1-\varepsilon^j)\Big\}$$
$$= \sum_{j=1}^{k-1}\log\Big(2\sin\frac{\pi j}{k}\Big)\cos\frac{2\pi rj}{k}+\frac{\pi}{2k}\sum_{j=1}^{k-1}(2j-k)\sin\frac{2\pi rj}{k}.$$

Applying Lemma A (a) and (c) to the second sum reduces it to

$$-\frac{\pi}{2}\cot\frac{\pi r}{k}.$$

By (b) the first sum in (10) is equal to

$$\sum_{j=1}^{k-1}\Big\{\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}\Big\}-\log 2.$$

In this sum the terms corresponding to j and $k-j$ are equal and when $j = k-j$ the term is zero. Hence we can finally write, when $r \not\equiv 0 \pmod{k}$,

$$(11)\quad k\gamma(r,k) = \gamma+\log 2+\frac{\pi}{2}\cot\frac{\pi r}{k}-2\sum_{0<j<k/2}\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}.$$

One useful and immediate consequence of (11) is the formula

$$(12)\qquad \gamma(k-r,k) = \gamma(r,k)-\frac{\pi}{k}\cot\frac{\pi r}{k}\qquad (r \not\equiv 0 \pmod{k}).$$

Of course (11) can be further simplified when k is a specific small integer. This matter will be discussed at the end of the next section.

4. Primitive $\gamma(r,k)$. When r and k are coprime we call $\gamma(r,k)$ *primitive*, otherwise *imprimitive*. It is clear that if $\gamma(r,k)$ is imprimitive it must be related to some $\gamma(r_1,k_1)$ with $k_1<k$. More precisely we have

THEOREM 2. *Let δ be any common divisor of r and k. Then*

$$\gamma(r,k) = \frac{1}{\delta}\gamma(r/\delta, k/\delta) - \frac{1}{k}\log\delta. \tag{13}$$

Remark. In case δ is the greatest common divisor (r,k) of r and k then $\gamma(r/\delta, k/\delta)$ is primitive. In case $r = 0$ then (13) becomes (2).

Proof. Let $r_1 = r/\delta$ and $k_1 = k/\delta$. Then

$$H(x,r,k) = \frac{1}{\delta}H(x/\delta, r_1, k_1)$$

and so

$$H(x,r,k) - \frac{1}{k}\log x = \frac{1}{\delta}\left\{H(x/\delta, r_1, k_1) - \frac{1}{k_1}\log x/\delta\right\} - \frac{1}{k}\log\delta.$$

Letting $x\to\infty$ gives the theorem.

We now consider the sum of all the $\varphi(k)$ primitive $\gamma(r,k)$ where k is fixed. We denote the sum by

$$\Phi(k) = \sum_{(r,k)=1}\gamma(r,k).$$

We derive an explicit formula for $\Phi(k)$, which depends on the prime factors of k, in a succession of three theorems. The first is

THEOREM 3. *There exists a rational number N_k such that*

$$k\Phi(k) = \varphi(k)\gamma + \log N_k.$$

Proof. The theorem is true for $k=1$ with $N_1 = 1$. Supposing the theorem holds for all the proper divisors of k. We can then write, in view of (3)

$$\Phi(k) = \gamma - \sum_{\substack{\delta\mid k\\ \delta>1}}\sum_{(r,k)=\delta}\gamma(r,k).$$

Applying Theorem 2 we have

$$\begin{aligned}
-k(\Phi(k)-\gamma) &= \sum_{\substack{\delta\mid k\\ \delta>1}}\sum_{(r,k)=\delta} k\left\{\frac{1}{\delta}\gamma\left(\frac{r}{\delta},\frac{k}{\delta}\right) - \frac{1}{k}\log\delta\right\}\\
&= \sum_{\substack{\delta\mid k\\ \delta>1}}\left\{\frac{k}{\delta}\Phi\left(\frac{k}{\delta}\right) - \Phi\left(\frac{k}{\delta}\right)\log\delta\right\}\\
&= \sum_{\substack{d\mid k\\ d<k}} d\Phi(d) - \sum_{d\mid k}\varphi(d)\log(k/d)\\
&= \sum_{\substack{d\mid k\\ d<k}}\{\varphi(d)\gamma + \log N_d\} - \log k\sum_{d\mid k}\varphi(d) + \sum_{d\mid k}\varphi(d)\log d.
\end{aligned}$$

Here we have used our induction hypothesis. Since

$$\sum_{d\mid k}\varphi(d) = k$$

we have

$$k\Phi(k) - \varphi(k)\gamma = k\log k - \sum_{d\mid k}\varphi(d)\log d - \sum_{\substack{d\mid k\\ d<k}}\log N_d. \tag{14}$$

Now the right hand side is evidently the logarithm of some rational number. Call it N_k since it depends only on k. This proves Theorem 3.

THEOREM 4. *The N_k of Theorem 3 is given by*

$$N_k = \prod_{\delta\mid k}\delta^{-k\mu(\delta)/\delta}$$

where μ is Möbius' function.

Proof. Let the numerical function f be defined by

$$f(k) = \sum_{\delta\mid k}\log N_\delta.$$

Then (14) can be written in view of Theorem 3,

$$k\log k - f(k) = \sum_{\delta\mid k}\varphi(\delta)\log\delta.$$

By Möbius inversion

$$\varphi(k)\log k = \sum_{\delta\mid k}\delta\log\delta\,\mu(k/\delta) - \sum_{\delta\mid k}f(\delta)\mu(k/\delta).$$

Again by Möbius inversion the second sum is simply $\log N_k$. Thus we have

$$\begin{aligned}
\log N_k &= \sum_{\delta\mid k}\delta\log\delta\,\mu(k/\delta) - \log k\sum_{\delta\mid k}(k/\delta)\mu(\delta)\\
&= \sum_{\delta\mid k}\mu(\delta)\frac{k}{\delta}(\log k - \log\delta) - k\log k\sum_{\delta\mid k}\mu(\delta)/\delta\\
&= -k\sum_{\delta\mid k}\big(\mu(\delta)/\delta\big)\log\delta.
\end{aligned}$$

Taking exponentials gives the theorem.

THEOREM 5. *N_k is, in fact, the integer*

$$N_k = \prod_{p\mid k}p^{\varphi(k)/(p-1)}$$

where p ranges over the prime factors of k.

Proof. By Theorem 4 it is clear that N_k is the product of powers (positive, zero or negative) of the prime factors of k. Let p be any one of these primes and let

$$k = p^a m \quad (p\nmid m),$$
$$N_k = p^\beta n \quad (p\nmid n).$$

We must show that

$$\beta = \varphi(k)/(p-1). \tag{15}$$

Any δ dividing k is of the form $\delta = p^i d$ where $d\,|\,m$. By Theorem 4

$$\beta = -\sum_{d|m}\sum_{j=0}^{a} j m p^{a-j}\mu(p^j)\mu(d)/d.$$

It is clear that we must take $j=1$ to get any contribution to β. Hence

$$\beta = p^{a-1}\sum_{d|m}\frac{m}{d}\mu(d) = p^{a-1}\varphi(m) = \varphi(p^a)\varphi(m)/(p-1) = \varphi(k)/(p-1)$$

which is (15).

Theorems 3 and 5 now yield

$$\Phi(k) = \frac{\varphi(k)}{k}\left(\gamma+\sum_{p|k}\frac{\log p}{p-1}\right). \tag{16}$$

As one consequence of (16), $\Phi(k)$ depends only on the prime factors of k. In fact if δ is a divisor of k then $\Phi(\delta k) = \Phi(k)$. As another application of (16) we have

THEOREM 6. *If* $k>1$

$$2\sum_{0<j<k/2}\frac{\mu\big(k/(k,j)\big)}{\varphi\big(k/(k,j)\big)}\log\sin\frac{\pi j}{k} = \log 2-\sum_{p|k}\frac{\log p}{p-1}.$$

Proof. If we sum both sides of (11) over the $\varphi(k)$ numbers r which are $<k$ and prime to k we obtain

$$k\Phi(k) = (\gamma+\log 2)\varphi(k)-\sum_{0<j<k/2}\log\sin\frac{\pi j}{k}\sum_{(r,k)=1}2\cos\frac{2\pi rj}{k}$$

since the cotangent terms for r and $k-r$ destroy themselves.

Now the inner sum is

$$2\sum_{(r,k)=1}e^{2\pi i jr/k} = 2c_k(j) = 2\varphi(k)\frac{\mu\big(k/(k,j)\big)}{\varphi\big(k/(k,j)\big)}$$

which is a well-known formula of Hölder for the Ramanujan sum $c_k(j)$ [5]. Substituting for $\Phi(k)$ from (16) and simplifying gives the theorem.

In case $k=p$, a prime, Theorem 6 becomes the familiar result

$$\prod_{r=1}^{p-1}2\sin\frac{\pi r}{p} = p.$$

Theorem 6 is an instance of the catalytic effect of the Euler constants $\gamma(r,k)$. Further examples occur in § 6.

It is well-known [8] that the values of the trigonometric functions in formula (11) are algebraic numbers. In case k is a power of 2 times a (possibly empty) product of distinct primes of the form $2^{2^\lambda}+1$ these algebraic numbers are expressible in terms of successive square roots of positive integers, that is they are constructable with ruler and compass.

We give below a condensed list of $\gamma(r,k)$ expressed in this way. To save printing costs we can omit the imprimitive cases by Theorem 2. We can also dispense with the cases in which $k\equiv 2 \pmod 4$ since by (5) and Theorem 2

$$\gamma(2s+1,4m+2) = \gamma(2s+1,2m+1)-\tfrac{1}{2}\gamma(s+m+1,2m+1)+\frac{\log 2}{4m+2}.$$

Finally we can suppose that $r<k/2$ in view of (12).

5. $\gamma(r,k)$ **and** $\Gamma'(z)/\Gamma(z)$. From the three familiar basic formulas

$$\Gamma(1+z) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1-z) = \pi\csc\pi z,$$
$$1/\Gamma(1+z) = e^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-z/n}\right\}$$

for the Gamma function the following well-known properties of its logarithmic derivative $\psi(z) = \Gamma'(z)/\Gamma(z)$ are immediately derived.

$$\psi(1+z) = \psi(z)+1/z, \tag{17}$$

$$\psi(1-z) = \psi(z)+\pi\cot\pi z, \tag{18}$$

$$\psi(1+z) = -\gamma+z\sum_{n=1}^{\infty}\frac{1}{n(n+z)} = -\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1}. \tag{19}$$

The following theorem indicates a connection between $\gamma(r,k)$ and $\psi(z)$ at the rational point r/k.

THEOREM 7.

$$\gamma(r,k) = -\{\psi(r/k)+\log k\}/k \quad (0<r\leqslant k).$$

LIST OF CERTAIN CONSTRUCTABLE $\gamma(r,k)$

$$\gamma(1,2)=\frac{1}{2}\gamma+\frac{1}{2}\log 2,$$

$$\gamma(1,3)=\frac{1}{3}\gamma+\frac{\pi}{18}\sqrt{3}+\frac{1}{6}\log 3,$$

$$\gamma(1,4)=\frac{1}{4}\gamma+\frac{\pi}{8}+\frac{1}{4}\log 2,$$

$$\gamma(1,5)=\frac{1}{5}\gamma+\frac{\pi}{10}\sqrt{1+2/\sqrt{5}}+\frac{1}{20}\log 5+\frac{\sqrt{5}}{10}\log(1+\sqrt{5})/2,$$

$$\gamma(2,5)=\frac{1}{5}\gamma+\frac{\pi}{10}\sqrt{1-2/\sqrt{5}}+\frac{1}{20}\log 5-\frac{\sqrt{5}}{10}\log(1+\sqrt{5})/2,$$

$$\gamma(1,8)=\frac{1}{8}\gamma+\frac{1}{8}\left\{\frac{\pi}{2}(\sqrt{2}+1)+\log 2+\sqrt{2}\log(\sqrt{2}+1)\right\},$$

$$\gamma(3,8)=\frac{1}{8}\gamma+\frac{1}{8}\left\{\frac{\pi}{2}(\sqrt{2}-1)+\log 2-\sqrt{2}\log(\sqrt{2}+1)\right\},$$

$$\gamma(1,12)=\frac{1}{12}\gamma+\frac{1}{24}\{\pi(2+\sqrt{3})-2(\sqrt{3}-1)\log 2+\log 3+4\sqrt{3}\log(\sqrt{3}+1)\},$$

$$\gamma(5,12)=\frac{1}{12}\gamma+\frac{1}{24}\{\pi(2-\sqrt{3})+2(\sqrt{3}+1)\log 2+\log 3-4\sqrt{3}\log(\sqrt{3}+1)\}.$$

The small Table 1 giving $\gamma(r,k)$ to $6D$ for all r and for $k<10$ may be helpful in checking formulas and in applications.

Table 1

r	k	$\gamma(r,k)$	r	k	$\gamma(r,k)$	r	k	$\gamma(r,k)$
1	1	.577216	1	6	.756728	3	8	.084320
1	2	.635182	2	6	.223379	4	8	−.014491
2	2	−.057966	3	6	.028625	5	8	−.078342
1	3	.677807	4	6	−.078921	6	8	−.124198
2	3	.073207	5	6	−.150172	7	8	−.159428
3	3	−.173799	6	6	−.202424	8	8	−.187778
1	4	.710290	1	7	.774010	1	9	.801191
2	4	.144304	2	7	.248515	2	9	.284890
3	4	−.075108	3	7	.059987	3	9	.103868
4	4	−.202270	4	7	−.042448	4	9	.007727
1	5	.735920	5	7	−.109390	5	9	−.053823
2	5	.190389	6	7	−.157931	6	9	−.097666
3	5	−.013764	7	7	−.195528	7	9	−.131110
4	5	−.128886	1	8	.788631	8	9	−.157860
5	5	−.206444	2	8	.268501	9	9	−.180001

Proof. By definition and by (17) and (19)

$$\gamma(r,k)=\lim_{x\to\infty}\left\{\frac{1}{r}+\frac{1}{k+r}+\frac{1}{2k+r}+\ldots+\frac{1}{\left[\frac{x}{k}\right]k+r}-\frac{1}{k}\log x\right\}$$

$$=\frac{1}{r}+\lim_{x\to\infty}\left\{\frac{1}{k}-\left(\frac{1}{k}-\frac{1}{k+r}\right)+\ldots+\frac{1}{\left[\frac{x}{k}\right]k}-\left(\frac{1}{\left[\frac{x}{k}\right]k}-\frac{1}{\left[\frac{x}{k}\right]k+r}\right)-\frac{1}{k}\left(\log\frac{x}{k}+\log k\right)\right\}$$

$$=\frac{1}{r}+\frac{1}{k}\gamma-\frac{\log k}{k}-\sum_{n=1}^{\infty}\frac{r}{nk(nk+r)}=\frac{1}{k}\left\{\frac{k}{r}-\log k+\gamma-\frac{r}{k}\sum_{n=1}^{\infty}\frac{1}{n(n+r/k)}\right\}$$

$$=\frac{1}{k}\left\{-\log k+\frac{k}{r}-\psi\left(1+\frac{r}{k}\right)\right\}=\frac{1}{k}\{-\log k-\psi(r/k)\}$$

which is the theorem.

Solving for $\psi(r/k)$ we find

$$\psi(r/k)=-\{k\gamma(r,k)+\log k\}\qquad(0<r\leqslant k)\tag{20}$$

which, as we note, holds also for $r=k$.

The results we have already obtained for $\gamma(r,k)$ can now be applied to give information about $\psi(z)$. For example (1) gives us at once

$$\psi(r/k)=-\gamma-\log(k/2)-\frac{\pi}{2}\cot\frac{\pi r}{k}+2\sum_{0<j<k/2}\cos\frac{2\pi rj}{k}\log\sin\frac{\pi j}{k}.$$

This was discovered by Gauss in 1813 [4]. A simplification of Gauss' proof has been given by Jensen [6] using Abel's theorem on the continuity of convergent power series on the circle of convergence. Our proof via finite Fourier series indicates that Gauss' remarkable result has a completely elementary basis. Gauss used this result to produce the first table of $\psi(z)$ for $z=0(.01)1$. He also pointed out that because of (17) and (18) we can evaluate $\psi(z)$ at any rational point $\neq 0$. For example we have

$$\psi(3/4)=-\gamma+\frac{\pi}{2}-\log 8,$$

$$\psi(7/3)=\frac{15}{4}-\frac{3}{2}\log 3-\gamma-\frac{\pi\sqrt{3}}{6},$$

$$\psi(-3/2)=\frac{8}{3}-\gamma-\log 4.$$

The relation (3) gives us

$$\sum_{r=1}^{k-1} \psi(r/k) = -\{k\log k+(k-1)\gamma\}$$

while from (18) we obtain

$$\sum_{\substack{r=1\\(r,k)=1}}^{k-1} \psi(r/k) = -\varphi(k)\left\{\gamma+\log k+\sum_{p|k}\frac{\log p}{p-1}\right\}.$$

6. Applications of $\gamma(r, k)$. As a first application we give

THEOREM 8. *Let $g(n)$ be a numerical function which is purely periodic of period k. Then*

$$S(g) \equiv \sum_{n=1}^{\infty} g(n)/n = \sum_{r=1}^{k} g(r)\gamma(r, k)$$

provided $\sum_{r=1}^{k} g(r) = 0$; which is a necessary and sufficient condition for convergence of $S(g)$.

Proof. We have

$$\sum_{1\leqslant n\leqslant x} g(n)/n = \sum_{r=1}^{k} g(r)H(x, r, k)$$
$$= \sum_{r=1}^{k} g(r)\left\{H(x, r, k)-\frac{1}{k}\log x\right\}+\frac{\log x}{k}\sum_{r=1}^{k} g(r).$$

As $x\to\infty$ the first sum tends to $\sum g(r)\gamma(r, k)$. The second sum must vanish for convergence.

Theorem 8 can be used to evaluate $S(g)$ in terms of the $\gamma(r, k)$ or inversely, when $S(g)$ is known, to obtain interesting linear combinations of the $\gamma(r, k)$. To illustrate the latter use we can choose $g(n) = \varepsilon^{nj}$ where $j\not\equiv 0 \pmod k$ and $\varepsilon = e^{2\pi i/k}$. Then $S(g)$ becomes $-\log(1-\varepsilon^j)$ and we obtain (9). To illustrate the former use, consider the example

$$g(1) = g(2) = \ldots = g(k-1) = 1, \quad g(k) = 1-k.$$

This gives us the series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k-1}+\frac{1-k}{k}+\frac{1}{k+1}+\ldots+\frac{1}{2k-1}+\frac{1-k}{2k}+\ldots = \log k$$

using (3) and (2). This is a generalization of

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots = \log 2.$$

A more interesting and conspicuous class of examples of $S(g)$ are the Dirichlet L series $L(s, \chi)$ at $s = 1$ namely

$$L(1, \chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n}$$

where $\chi(n)$ is a non-principle character modulo k, that is, a non-constant purely multiplicative periodic function of period k which vanishes whenever n and k have a prime factor in common. For n prime to k, $\chi(n)$ is a $\varphi(k)$-th root of unity, not necessarily primitive. Of particular interest to the theory of quadratic fields are the cases in which k or $-k$ is the discriminant of a monic irreducible quadratic equation, because of the connection between $L(1, \chi)$ and the class number $h(\sqrt{\pm k})$ of the field of that equation. For us this means that the two linear combinations

$$\sum_{r=1}^{k-1}\left(\frac{k}{r}\right)\gamma(r, k) \quad\text{and}\quad \sum_{r=1}^{k-1}\left(\frac{-k}{r}\right)\gamma(r, k),$$

where the symbols are those of Kronecker, can be expressed in terms of $h(\sqrt{\pm k})$. For $k>1$ the precise formulas are

$$\sum_{r=1}^{k-1}\left(\frac{k}{r}\right)\gamma(r, k) = 2\log\varepsilon_k k^{-1/2}h(\sqrt{k}), \tag{21}$$

$$\sum_{r=1}^{k-1}\left(\frac{-k}{r}\right)\gamma(r, k) = \frac{2\pi}{w}k^{-1/2}h(\sqrt{-k}) \tag{22}$$

where $w = 6$ for $k = 3$, $w = 4$ for $k = 4$ and $w = 2$ for all other k and ε_k is the fundamental unit in the real field $Q(\sqrt{k})$.

Thus for $k = 5$, (21) becomes

$$\gamma(1, 5)-\gamma(2, 5)-\gamma(3, 5)+\gamma(4, 5) = 2\log\left(\frac{1+\sqrt{+5}}{2}\right)/\sqrt{5}$$

and for $k = 12$, (22) becomes

$$\gamma(1, 12)-\gamma(5, 12)+\gamma(7, 12)-\gamma(11, 12) = \pi/\sqrt{12}$$

since in both cases $h = 1$. These relations are easily verified from our list of $\gamma(r, k)$.

If in (21) and (22) we substitute for $\gamma(r, k)$ from (11) we obtain, after simplification, a pair of class number formulas for positive and negative discriminants.

These are

$$\varepsilon_k^{h(\sqrt{k})} = \frac{\prod\limits_N \sin(\pi N/k)}{\prod\limits_R \sin(\pi R/k)},$$

$$2\sqrt{k}\, h(\sqrt{-k}) = w \sum_r \left(\frac{-k}{r}\right)\cot(\pi r/k).$$

In the first relation, which is due to Dirichlet [3], N and R range over integers $\leqslant k/2$ for which $\left(\frac{k}{N}\right) = -1$ and $\left(\frac{k}{R}\right) = +1$. In the second relation, which is due to V. A. Lebesgue [7], r ranges over the positive integers $\leqslant k/2$.

As a corollary to Theorem 8 we can evaluate a more general sum.

COROLLARY. *Let a and b be relatively prime integers with $0 < a < b$ and let $g(n)$ be any numerical function periodic of period k. Then*

$$S(g, a, b) \equiv \sum_{n=0}^{\infty} \frac{g(n)}{n + a/b} = b \sum_{r=0}^{k-1} g(r)\gamma(br + a, bk)$$

provided $\sum_{r=1}^{k} g(r) = 0$, which is necessary and sufficient for convergence of $S(g, a, b)$.

Proof. Define $g_1(n)$ by

$$g_1(n) = \begin{cases} bg(n) & \text{if } n \equiv a (\operatorname{mod} b), \\ 0 & \text{otherwise.} \end{cases}$$

Now we can apply Theorem 8 with k replaced by bk.

Another type of sum that can be evaluated by $\gamma(r, k)$ is considered in

THEOREM 9. *Let $m \geqslant 2$ and let*

$$(r_1, k_1), (r_2, k_2), \ldots, (r_m, k_m)$$

be pairs of positive integers for which

$$0 < r_j \leqslant k_j \quad (j = 1(1)m)$$

and the m rational numbers r_j/k_j are distinct. Finally let $p(x)$ be any polynomial of degree $\leqslant m-2$, a necessary condition for convergence.

Then

$$S = \sum_{n=0}^{\infty} \frac{p(n)}{(k_1 n + r_1)(k_2 n + r_2) \ldots (k_m n + r_m)} = \sum_{j=1}^{m} c_j \left\{\gamma(r_j, k_j) + \frac{\log k_j}{k_j}\right\}$$

where the coefficients c_j are defined by the partial fraction decomposition

$$\frac{p(x)}{(k_1 x + r_1)(k_2 x + r_2) \ldots (k_m x + r_m)} = \sum_{j=1}^{m} \frac{c_j}{k_j x + r_j}. \tag{23}$$

Proof. By (23)

$$p(x) = \sum_{j=1}^{m} c_j \frac{(k_1 x + r_1) \ldots (k_m x + r_m)}{k_j x + r_j}.$$

The right hand side is a formal polynomial of degree $m-1$. Hence its leading coefficient must vanish. That is

$$\sum_{j=1}^{m} c_j/k_j = 0. \tag{24}$$

Again by (23)

$$S = \lim_{x\to\infty} \sum_{n \leqslant x} \frac{p(n)}{(k_1 n + r_1) \ldots (k_m n + r_m)}$$

$$= \lim_{x\to\infty} \sum_{j=1}^{m} c_j \sum_{0 \leqslant n \leqslant x} \frac{1}{k_j n + r_j} = \lim_{x\to\infty} \sum_{j=1}^{m} c_j H(k_j x, r_j, k_j)$$

$$= \sum_{j=1}^{m} c_j \lim_{x\to\infty} \left\{H(k_j x, r_j, k_j) - \frac{1}{k_j}\log k_j x\right\} +$$

$$+ \lim_{x\to\infty} \left\{(\log x) \sum_{j=1}^{m} c_j/k_j\right\} + \sum_{j=1}^{m} \frac{c_j}{k_j}\log k_j$$

$$= \sum_{j=1}^{m} c_j \left\{\gamma(r_j, k_j) + \frac{1}{k_j}\log k_j\right\}$$

in view of (24). This gives the theorem.

To give a simple illustration consider the sum

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)}.$$

Here we have

$$c_1 = \tfrac{1}{3}, \quad c_2 = -2, \quad c_3 = \tfrac{8}{3}$$

and so the sum is

$$S = \tfrac{1}{3}\gamma(1,1) - 2\{\gamma(1,2) + \tfrac{1}{2}\log 2\} + \tfrac{8}{3}\{\gamma(1,4) + \tfrac{1}{4}\log 4\}.$$

This reduces to $\pi/3 = 1.04719755\ldots$

Because of (24), those sums S for which the k_j are all equal will have the value

$$\sum_{j=1}^{m} c_j \gamma(r_j, k).$$

As an example we may cite the sum

$$S_k = \sum_{n=0}^{\infty} \frac{1}{(kn+1)(kn+2)\ldots(kn+k)} = \left(\sum_{r=1}^{k}(-1)^{r-1}\binom{k-1}{r-1}\gamma(r,k)\right)/(k-1)!.$$

For example when $k = 6$ we find

$$S_6 = \{192\log 2 - 81\log 3 - 7\pi\sqrt{3}\}/4320 = .01390480727.$$

The reader will have observed that in all these applications Euler's constant γ cancels out.

7. Numerical evaluation of $\gamma(r, k)$**.** Formula (11) which gives the exact value of $\gamma(r,k)$ is somewhat unwieldy and expensive for the numerical calculation of $\gamma(r,k)$ expecially for large k. If one has access to a good table of $\psi(1+z)$ such as [1] or [2] one can use the formula of Theorem 7

$$\gamma(r,k) = \frac{1}{r} - \frac{\log k}{k} - \frac{1}{k}\psi(1+r/k).$$

Alternatively one can use the series (19) instead of a table.

For automatic computing and where greater accuracy is desired one can apply the Euler-Maclaurin summation formula. This method avoids the use of trigonometric functions and allows one to "wholesale" the computation of $\gamma(r,k)$ for k fixed.

If in the Euler–Maclaurin formula

$$f(0)+f(1)+\ldots+f(N) = \int_0^N f(t)\,dt + \frac{1}{2}\{f(N)+f(0)\} +$$
$$+\frac{1}{12}\{f'(N)-f'(0)\} - \frac{1}{720}\{f'''(N)-f'''(0)\} +$$
$$+\frac{1}{30240}\{f^{(5)}(N)-f^{(5)}(0)\}\ldots$$

we set

$$f(t) = (kt+r)^{-1}, \quad kN+r = x$$

we obtain the asymptotic formula

$$(25)\quad \gamma(r,k) = H(x,r,k) - \frac{1}{k}\log x - \frac{1}{2x} + \frac{k}{12x^2} - \frac{k^3}{120x^4} + \frac{k^5}{252x^6} - \ldots$$

If we choose x near to 1000 and $k < 100$ these six terms alone give an error less than $5\cdot 10^{-13}$ in absolute value; for small k it is, of course, very much smaller.

For fixed k we can give a polynomial approximation to $\gamma(r,k)$ once the values of $H(1000, r, k)$ have been computed. To this effect coefficients C_i are defined as follows.

$$C_0 = \frac{1}{k}\log 1000 + \frac{1}{2\cdot 10^3} - \frac{k}{12\cdot 10^6} + \frac{k^3}{12\cdot 10^{13}} - \frac{k^5}{252\cdot 10^{18}},$$

$$C_1 = \frac{1}{k\cdot 10^3} - \frac{1}{2\cdot 10^6} + \frac{k}{6\cdot 10^9} - \frac{k^3}{3\cdot 10^6},$$

$$C_2 = \frac{1}{2k\cdot 10^6} - \frac{1}{2\cdot 10^9} + \frac{k}{4\cdot 10^{12}},$$

$$C_3 = \frac{1}{3k\cdot 10^9} - \frac{1}{2\cdot 10^{12}},$$

$$C_4 = \frac{1}{4k\cdot 10^{12}}.$$

Then from (25)

$$\gamma(r,k) \cong H(1000, r, k) - C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4$$

where

$$t \equiv 1000 - r \pmod{k} \quad (0 \leqslant t < k).$$

Similar polynomials based on other limits than 1000 can be written down from (25) whose general term is $k^{2n-1}B_{2n}x^{-2n}$, B_{2n} being, of course, the Bernoulli number of index $2n$. Such formulas are sometimes useful in case k is large and only a few values of r are involved. Such an occasion is the following question.

Inspection of our modest table of $\gamma(r,k)$ leads one to guess that about half of the $\gamma(r,k)$ are positive. This is in reality far from the truth, as we see from

THEOREM 10. *For large k the monotone sequence*

$$\gamma(1,k), \gamma(2,k), \ldots, \gamma(k,k)$$

changes sign in the neighborhood of $r = k/\log k$, so that almost all $\gamma(r,k)$ are negative.

Proof. We shall prove somewhat more. Let $L = \log k - \gamma$. By Theorem 7 to make $\gamma(r, k) = 0$ we must have

$$\psi(z) = -\log k, \quad z = r/k$$

or, by (17),

(26) $$z\psi(1+z) = 1 - z\log k.$$

But by (19)

$$z\psi(1+z) = -\gamma z + \sum_{n=2}^{\infty} \zeta(n)(-z)^n.$$

Therefore (26) becomes

(27) $$z = L^{-1}\{1 - \zeta(2)z^2 + \zeta(3)z^3 - \ldots\}.$$

Taking only the first term the theorem follows. Solving (27) by iteration we find

(28) $$z = L^{-1} - \zeta(2)L^{-3} + \zeta(3)L^{-4} + [2\{\zeta(2)\}^2 - \zeta(4)]L^{-5} - \\ - [5\zeta(2)\zeta(3) + \zeta(5)]L^{-6} - \ldots$$

As an illustration we take $k = 100$. The terms of (28) become

$$.2483 - .0252 + .0046 + .0041 - .0025 = .2292.$$

By actual computation we find

$$\gamma(22, 100) = .00204268,$$
$$\gamma(23, 100) = -.00056747.$$

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On the distribution of additive arithmetic functions

by

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Dedicated to the memory of Yu. V. Linnik

Let $g(n)$ be a real valued additive arithmetic function (i.e. $g(mn) = g(m)+g(n)$ if $(m, n) = 1$). The distribution of values of such functions has been extensively investigated. As a new direction, Erdös, Ruzsa and Sárközi [1] proposed to estimate

(1) $$\max_{-\infty<a<\infty} N(a, x) \overset{\text{def}}{=} \max_{-\infty<a<\infty} \sum_{\substack{n\leqslant x \\ g(n)=a}} 1$$

for general additive functions. They found bounds cx in various cases, often giving the best possible value of c. If, however, $g(n) = \omega(n)$, the number of prime divisors of n, then this quantity is about const $\frac{x}{\sqrt{\log\log x}}$ and they conjectured (oral communication) that this order of magnitude cannot be exceeded in any case, provided that $g(p) \neq 0$ for each prime p. The aim of this paper is to prove this conjecture in the following more precise form.

THEOREM. *Let $g(n)$ be an arbitrary real valued additive function and put*

$$E(x) = \sum_{\substack{p\leqslant x \\ g(p)\neq 0}} \frac{1}{p}.$$

Then there is a universal constant c_1 such that

$$N(a, x) = \sum_{\substack{n\leqslant x \\ g(n)=a}} 1 \leqslant c_1 \frac{x}{\sqrt{E(x)}}.$$

The result is sharp even in this more general form: The bound is attained if $g(p) = 0$ or 1 and $\sum_{\substack{p\leqslant x \\ g(p)=1}} 1/p = E(x)$ as is seen from [2] and [3] where much more detailed information is given in this special case. (For refer-